# A New Interior Schwarzschild Solution on the Gravitational Field of a Sphere of Compressible Fluid in General Relativity 

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#### Abstract

A new interior Schwarzschild solution is presented. It is static, spherically symmetric, regular everywhere inside a sphere of radius $r_{1}$, and across the surface of this sphere it is joined smoothly to the exterior Schwarzschild solution. There are no radial stresses inside the sphere. The radius $r_{1}$ is subject to the inequality $r_{1}>2 m$, where $m$ is the gravitational mass of the sphere. Under certain conditions the new solution may be interpreted as the field inside an Einstein cluster.


Keywords: space-time metric, Einstein-Maxwell equations, Energy- momentum tensor, Compressible fluid sphere

## 1. Introduction

The well known Schwarzschild interior solution, representing the field of a fluid sphere of constant density $\rho$, was discovered more than ninety years ago and still holds a prominent place in relativity theory. Schwarzschild obtained a solution for a sphere of homogeneous incompressible fluid [1]. His solution has been generalized for an infinite class of equivalent metrics [2]. These solutions demonstrate that there is an upper bound and a lower bound on the size of a sphere of homogeneous incompressible fluid that can exist. There is a solution for a fluid sphere of constant gravitational mass density $\rho+3 p / c^{2}$. As was shown by Whittaker [4], it is the expression rather than $\rho$ which governs the gravitational attraction of matter, p denotes the pressure. The type of general relativistic model which is discussed here is necessary for very dense stars such as white dwarves and neutron stars where the densities are extremely high. Utilizing Schwarzschild's particular solution [3] we shall extend his result to a general solution for a sphere of compressible fluid. At the surface of the sphere the required solution must maintain a smooth transition from the field outside the sphere to the field inside the sphere. Therefore, the metric for the interior and the metric for the exterior must attain the same value for the radius of curvature at the surface of the sphere.

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## 2. The general interior solution for Schwarzschild's incompressible sphere of fluid

To determine the line element inside the sphere of matter, the solution of Einstein's field equation must depend on the properties of fluid of which the sphere is composed. Schwarzschild solved this problem by assuming that the sphere is composed of an incompressible perfect fluid of proper density $\rho$ and proper pressure $P$.

The space-time in the interior of a charged fluid sphere in equilibrium is appropriately described by the metric

$$
\begin{equation*}
d s^{2}=c^{2} b(r) d t^{2}-a(r) d r^{2}-r^{2}\left(d \theta^{2}+\sin ^{2} \theta d \varphi^{2}\right) \tag{2.1}
\end{equation*}
$$

The metric potentials $\mathrm{b}(\mathrm{r})$ and $\mathrm{a}(\mathrm{r})$ are governed by the coupled Einstein Maxwell equations

$$
\begin{equation*}
g^{\alpha \mu} R_{\alpha v}-\frac{1}{2} \delta_{v}^{\mu} R-\Lambda \delta_{v}^{\mu}=-\mathrm{K} \mathrm{~T}^{\mu}{ }_{v} \tag{2.2}
\end{equation*}
$$

where the energy-momentum tensor describing the physical content of the space-time is given by

$$
\begin{equation*}
T^{\mu}{ }_{v}=(P+\rho) g_{v \alpha} u^{\mu} u^{\alpha}-P \delta^{\mu}{ }_{v} \tag{2.3}
\end{equation*}
$$

Since distribution of mass is static, all velocity components of fluid matter must be zero.

$$
\begin{aligned}
\mathrm{u}^{1}=\mathrm{u}^{2} & =\mathrm{u}^{3}=0 \text { and } \mathrm{u}^{0}=\frac{d t}{d s}=\frac{1}{c \sqrt{b}} \\
\therefore \quad \mathrm{~T}_{0}^{0} & =(\mathrm{P}+\rho) \mathrm{g}_{00} \mathrm{u}^{0} \mathrm{u}^{0}-\mathrm{P} \mathrm{\delta}^{0}{ }_{0}=(\mathrm{P}+\rho) \mathrm{c}^{2} \mathrm{~b} \frac{1}{c^{2} b}-\mathrm{P}=\rho \\
\mathrm{T}_{1}^{1} & =(\mathrm{P}+\rho) \mathrm{g}_{11} \mathrm{u}^{1} \mathrm{u}^{1}-\mathrm{P} \mathrm{\delta}^{1}{ }_{1}=-\mathrm{P} \\
\mathrm{~T}_{2}^{2} & =(\mathrm{P}+\rho) \mathrm{g}_{22} \mathrm{u}^{2} \mathrm{u}^{2}-\mathrm{P} \mathrm{\delta}^{2}{ }_{2} \\
\mathrm{~T}_{3}{ }_{3} & =(\mathrm{P}+\rho) \mathrm{g}_{33} \mathrm{u}^{3} \mathrm{u}^{3}-\mathrm{PS}_{3}{ }_{3}
\end{aligned}=-\mathrm{P} .
$$

Now, from equation (2.2)

$$
\begin{align*}
& \mathrm{g}^{00} \mathrm{R}_{00}-1 / 2 \mathrm{R}+\Lambda=-\mathrm{K} \mathrm{~T}_{0}^{0} \\
& \text { or, } \frac{1}{a r^{2}}-\frac{a^{\prime}}{a^{2} r}-\frac{1}{r^{2}}+\Lambda=-\mathrm{K} \rho  \tag{2.4}\\
& \text { or, } \frac{b^{\prime}}{a b r}+\frac{1}{a r^{2}}-\frac{1}{r^{2}}+\Lambda=\mathrm{KP} \\
& \mathrm{~g}^{22} \mathrm{R}_{22}-1 / 2 \delta^{2}{ }_{2} \mathrm{R}+\Lambda \delta^{2}{ }_{2}=-\mathrm{K} \mathrm{~T}^{2}{ }_{2}  \tag{2.5}\\
& \text { or, }-\frac{a^{\prime}}{2 a^{2} r}+\frac{b^{\prime}}{2 a b r}-\frac{a^{\prime} b^{\prime}}{4 a^{2} b}+\frac{b^{\prime \prime}}{2 a b}-\frac{b^{\prime 2}}{4 a b^{2}}+\Lambda=\mathrm{KP} \\
& \mathrm{~g}^{33} \mathrm{R}_{33}-1 / 2 \delta^{1} \delta_{3}^{3} \mathrm{R}+\Lambda \delta_{3}^{3}=-\mathrm{K} \mathrm{~T}_{1}^{1}=-\mathrm{K} \mathrm{~T}_{3}^{3}  \tag{2.6}\\
& \text { or, }-\frac{a^{\prime}}{2 a^{2} r}+\frac{b^{\prime}}{2 a b r}-\frac{a^{\prime} b^{\prime}}{4 a^{2} b}+\frac{b^{\prime \prime}}{2 a b}-\frac{b^{\prime 2}}{4 a b^{2}}+\Lambda=\mathrm{KP}
\end{align*}
$$

Evidently (2.6) and (2.7) are identical. Hence we have three independent equations namely (2.4), (2.5) and (2.6)

From equation (2.4)

$$
\begin{gather*}
\frac{1}{a r^{2}}-\frac{1}{a^{2} r} \frac{d a}{d r}=\frac{1}{r^{2}}-\mathrm{K} \rho-\Lambda \\
\text { or, } d\left(\frac{r}{a}\right)=\left[1-(\mathrm{K} \rho+\Lambda) \mathrm{r}^{2}\right] \mathrm{dr}  \tag{2.8}\\
\text { or, } \quad \frac{r}{a}=\mathrm{r}-\frac{\mathrm{K} \rho+\Lambda}{3} \mathrm{r}^{3}+\alpha
\end{gather*}
$$

where $\alpha$ is a constant of integration

$$
\therefore \quad \frac{1}{a}=1-\frac{K \rho+\Lambda}{3} r^{2}+\frac{\alpha}{r}
$$

Since at $\mathrm{r} \rightarrow 0, \frac{\alpha}{\mathrm{r}} \rightarrow \infty$, therefore to remove the singularity at $\mathrm{r}=0$, we put $\alpha=0$. Thus we have

$$
\begin{aligned}
& & \frac{1}{a} & =1-\frac{\mathrm{r}^{2}}{\mathrm{R}^{2}}, \quad \text { where } \frac{1}{\mathrm{R}^{2}}=\frac{\mathrm{K} \rho+\Lambda}{3} \\
\therefore & & a & =\left(1-\frac{r^{2}}{R^{2}}\right)^{-1}
\end{aligned}
$$

## Subtracting (2.4) from (2.5), we get

$$
\begin{equation*}
\frac{b^{\prime}}{a b r}+\frac{a^{\prime}}{a^{2} r}=K(P+\rho) \tag{2.9}
\end{equation*}
$$

Differentiating (2.5) with respect to r and then inserting (2.9), we get

$$
\begin{equation*}
\frac{\mathrm{b}^{\prime \prime}}{\mathrm{abr}}-\frac{2}{\mathrm{ar}^{3}}+\frac{2}{\mathrm{r}^{3}}=\mathrm{K} \frac{\mathrm{dP}}{\mathrm{dr}}+(\mathrm{P}+\rho) \mathrm{K}\left(\frac{\mathrm{~b}^{\prime}}{\mathrm{b}}+\frac{1}{\mathrm{r}}\right) \tag{2.10}
\end{equation*}
$$

Subtracting (2.5) from (2.6), we get

$$
-\frac{1}{2}\left(\frac{a^{\prime}}{a^{2} r}+\frac{b^{\prime}}{a b r}\right)-\frac{a^{\prime} b^{\prime}}{4 a^{2} b}-\frac{b^{\prime 2}}{4 a b^{2}}+\frac{b^{\prime \prime}}{2 a b}-\frac{1}{a r^{2}}+\frac{1}{r^{2}}=0
$$

Multiplying both sides by $\frac{2}{\mathrm{r}}$ and then inserting (2.9) and (2.10) we get,

$$
b=\left[A-B\left(1-\frac{r^{2}}{R^{2}}\right)^{1 / 2}\right]^{2}
$$

Now we have to determine the constants A and B.

$$
b^{\prime}=\frac{2 r B}{R^{2}}\left(1-\frac{r^{2}}{R^{2}}\right)^{\frac{-1}{2}}\left[\mathrm{~A}-\mathrm{B}\left(1-\frac{r^{2}}{R^{2}}\right)^{\frac{1}{2}}\right]
$$

Equation (2.5) becomes

$$
\frac{2 \mathrm{~B}}{\mathrm{R}^{2}} \frac{\left(1-\frac{\mathrm{r}^{2}}{\mathrm{R}^{2}}\right)^{\frac{1}{2}}}{\mathrm{~A}-\mathrm{B}\left(1-\frac{\mathrm{r}^{2}}{\mathrm{R}^{2}}\right)^{\frac{1}{2}}}-\frac{1}{\mathrm{R}^{2}}+\Lambda=\mathrm{KP}
$$

Using the boundary conditions
$\mathrm{P}=0$ at $\mathrm{r}=\mathrm{r}_{1}$, where $\mathrm{r}_{1}$ is the radius of the sphere and $\Lambda=0$ for $\mathrm{r} \leq \mathrm{r}_{1}$
we get,

$$
\begin{gathered}
\frac{2 B}{R^{2}} \frac{\left(1-\frac{r_{1}^{2}}{R^{2}}\right)^{\frac{1}{2}}}{A-B\left(1-\frac{r_{1}^{2}}{R^{2}}\right)^{\frac{1}{2}}}-\frac{1}{R^{2}}=0 \\
\text { or, } \quad A=3 B\left(1-\frac{r_{1}^{2}}{R^{2}}\right)^{\frac{1}{2}}
\end{gathered}
$$

The most celebrated exact solution of Einstein's vacuum field equations is the Schwarzschild exterior solution. In curvature coordinates $(r, \theta, \varphi, t)$ the solution takes the form

$$
\mathrm{ds}^{2}=\mathrm{c}^{2}\left(1-\frac{2 \mathrm{~m}}{\mathrm{r}}-\frac{\Lambda}{3} \mathrm{r}^{2}\right) \mathrm{dt} t^{2}-\left(1-\frac{2 \mathrm{~m}}{\mathrm{r}}-\frac{\Lambda}{3} \mathrm{r}^{2}\right)^{-1} \mathrm{dr}^{2}-\mathrm{r}^{2}\left(\mathrm{~d} \theta^{2}+\sin ^{2} \theta \mathrm{~d} \varphi^{2}\right)
$$

Taking $\mathrm{m}=0$, we obtain

$$
\mathrm{ds}^{2}=\mathrm{c}^{2}\left(1-\frac{\Lambda}{3} \mathrm{r}^{2}\right) \mathrm{dt} \mathrm{t}^{2}-\left(1-\frac{\Lambda}{3} \mathrm{r}^{2}\right)^{-1} \mathrm{dr}^{2}-\mathrm{r}^{2}\left(\mathrm{~d} \theta^{2}+\sin ^{2} \theta \mathrm{~d} \varphi^{2}\right)
$$

This is the Schwarzschild exterior solution for entirely empty world. This solution has a singularity at
$r=\sqrt{\frac{3}{\Lambda}}$. Since $\Lambda$ is very small, the value of $r$ is very large. It represents the horizon of the world. The exterior and interior solutions become identical at the boundary $r=r_{1}$ of the sphere.

$$
\begin{aligned}
& \therefore \quad 1-\frac{\mathrm{r}_{1}^{2}}{\mathrm{R}^{2}}=1-\frac{2 \mathrm{~m}}{\mathrm{r}_{1}} \\
& \text { or, } \quad \mathrm{m}=\frac{\mathrm{r}_{1}^{3}}{2 \mathrm{R}^{2}}
\end{aligned}
$$

Again, at $r=r_{1}$

$$
\begin{aligned}
& \frac{1}{\mathrm{R}^{2}}=\frac{\mathrm{K} \rho}{3} \quad[\text { since } \Lambda=0] \\
\therefore \quad & \mathrm{m}=\frac{\mathrm{K} \mathrm{\rho r}_{1}^{3}}{6}
\end{aligned}
$$

$$
\begin{array}{ll} 
& {\left[A-B\left(1-\frac{r_{1}^{2}}{R^{2}}\right)^{1 / 2}\right]^{2}=1-\frac{2 m}{r_{1}}} \\
\text { or, } & {\left[3 B\left(1-\frac{r_{1}^{2}}{R^{2}}\right)^{1 / 2}-B\left(1-\frac{r_{1}^{2}}{R^{2}}\right)^{1 / 2}\right]^{2}=1-\frac{2 m}{r_{1}}} \\
\text { or, } & \mathrm{B}=\frac{1}{2} \sqrt{\frac{1-\frac{2 \mathrm{~m}}{\mathrm{r}_{1}}}{1-\frac{\mathrm{r}_{1}^{2}}{\mathrm{R}^{2}}}} \\
\text { or, } & \mathrm{B}=\frac{1}{2} \sqrt{\frac{1-\frac{2}{r_{1}} \frac{r_{1}^{3}}{2 R^{2}}}{1-\frac{r_{1}^{2}}{R^{2}}}}=\frac{1}{2} \\
\therefore & \mathrm{~A}=\frac{3}{2}\left(1-\frac{\mathrm{r}_{1}^{2}}{\mathrm{R}^{2}}\right)
\end{array}
$$

Hence Schwarzschild's interior solution is

$$
\mathrm{ds}^{2}=c^{2}\left[A-B\left(1-\frac{r^{2}}{R^{2}}\right)^{1 / 2}\right]^{2} \mathrm{dt}^{2}-\left(1-\frac{r^{2}}{R^{2}}\right)^{-1} \mathrm{dr}^{2}-\mathrm{r}^{2}\left(\mathrm{~d} \theta^{2}+\sin ^{2} \theta \mathrm{~d} \varphi^{2}\right)
$$

The interior solution will be real only if

$$
\begin{array}{lll} 
& 1-\frac{\mathrm{r}_{1}^{2}}{\mathrm{R}^{2}}>0 \quad \text { and } & 1-\frac{2 \mathrm{~m}}{\mathrm{r}_{1}}>0 \\
\therefore & 2 \mathrm{~m}<\mathrm{r}_{1}<\sqrt{\frac{3}{\mathrm{~K} \rho}}
\end{array}
$$

This provides as upper limit on the possible size of a sphere of a given density and on the mass of a sphere of given radius.

## 3. A new interior Schwarzschild solution

If we consider the fluid be compressible, then we can assume

$$
\begin{equation*}
P=h \rho=\frac{\beta}{r^{2}} \tag{3.1}
\end{equation*}
$$

where $\beta$ and $h$ are constants.
The equation (2.8) becomes

$$
\begin{array}{ll} 
& d\left(\frac{r}{a}\right)=\left[1-\frac{K \beta}{h}-\Lambda r^{2}\right] d r \\
\text { or, } & \frac{r}{a}=\left(1-\frac{K \beta}{h}\right) r-\frac{\Lambda r^{3}}{3}+\gamma
\end{array}
$$

where $\gamma$ is the constant of integration.

$$
\therefore \quad a=\left(1-\frac{K \beta}{h}-\frac{\Lambda r^{2}}{3}+\frac{\gamma}{r}\right)^{-1}
$$

When $r \rightarrow 0, \frac{\gamma}{r} \rightarrow \infty$, therefore to remove the singularity at $\mathrm{r}=0$, we put $\gamma=0$.
Thus we have

$$
a=\left(1-\frac{K \beta}{h}-\frac{\Lambda r^{2}}{3}\right)^{-1}
$$

Again for the condition (3.1), equation (2.9) becomes

$$
\frac{b^{\prime}}{a b r}+\frac{a^{\prime}}{a^{2} r}=K\left(\frac{\beta}{r^{2}}+\frac{\beta}{h r^{2}}\right)
$$

or, $\frac{1}{b} \frac{d b}{d r}=\frac{\beta K\left(1+\frac{1}{h}\right) \frac{1}{r}-\frac{2 \Lambda r}{3}}{1-\frac{K \beta}{h}-\frac{\Lambda r^{2}}{3}}$
or,

$$
\begin{aligned}
& \frac{1}{b} \frac{d b}{d r}=\beta K\left(1+\frac{1}{h}\right)\left(1-\frac{K \beta}{h}\right)^{-1} \frac{1}{r}+\beta K\left(1+\frac{1}{h}\right) \frac{\Lambda}{3}\left(1-\frac{K \beta}{h}\right)^{-1} \frac{r}{1-\frac{K \beta}{h}-\frac{\Lambda r^{2}}{3}}-\frac{\frac{2 \Lambda r}{3}}{1-\frac{K \beta}{h}-\frac{\Lambda r^{2}}{3}} \\
& \therefore \quad b=\alpha r^{\beta K\left(1+\frac{1}{h}\right)\left(1-\frac{K \beta}{h}\right)^{-1}}\left(1-\frac{K \beta}{h}-\frac{\Lambda r^{2}}{3}\right)^{-\frac{1}{2} \beta K\left(1+\frac{1}{h}\right)\left(1-\frac{K \beta}{h}\right)^{-1}+1}
\end{aligned}
$$

where $\alpha$ is a constant of integration.
The boundary conditions are
$\mathrm{P}=0$ at $\mathrm{r}=\mathrm{r}_{1}$, where $\mathrm{r}_{1}$ is the radius of the sphere and $\Lambda=0$ for $\mathrm{r} \leq \mathrm{r}_{1}$
Now (3.1) implies that $r_{1} \rightarrow \infty$, which represents the horizon of the universe.

The exterior and interior solutions become identical at the boundary $r=r_{1}$ of the sphere.

$$
\therefore \quad a=\left(1-\frac{2 m}{r_{1}}-\frac{\Lambda r_{1}^{2}}{3}\right)^{-1}=\left(1-\frac{K \beta}{h}-\frac{\Lambda r_{1}^{2}}{3}\right)^{-1}
$$

Using the boundary conditions, we get

$$
1-\frac{2 m}{r_{1}}=1-\frac{K \beta}{h}
$$

or, $\quad \mathrm{m}=\frac{\mathrm{K} \beta \mathrm{r}_{1}}{2 h}$
and

$$
b=1-\frac{2 m}{r_{1}}-\frac{\Lambda r_{1}^{2}}{3}=\alpha r^{\beta K\left(1+\frac{1}{h}\right)\left(1-\frac{K \beta}{h}\right)^{-1}}\left(1-\frac{K \beta}{h}-\frac{\Lambda r^{2}}{3}\right)^{-\frac{1}{2} \beta K\left(1+\frac{1}{h}\right)\left(1-\frac{K \beta}{h}\right)^{-1}+1}
$$

Using the boundary conditions, we get for $\beta \mathrm{K}=-1, \beta>0$

$$
\alpha=\frac{r_{1}-2 m}{\left(1+\frac{1}{h}\right)^{3 / 2}}
$$

Hence, $\quad b=\frac{\alpha}{r}\left(1+\frac{1}{h}-\frac{\Lambda r^{2}}{3}\right)^{\frac{3}{2}}$
Therefore, the interior solution is

$$
d s^{2}=c^{2} \frac{\alpha}{r}\left(1+\frac{1}{h}-\frac{\Lambda r^{2}}{3}\right)^{\frac{3}{2}} d t^{2}-\left(1+\frac{1}{h}-\frac{\Lambda r^{2}}{3}\right)^{-1} d r^{2}-r^{2}\left(d \theta^{2}+\sin ^{2} \theta d \varphi^{2}\right)
$$

## 4. Conclusion

The new interior solution will be real only if $\mathrm{r}>0$ and $h \notin[-1,0]$. When $\mathrm{r}=0$, the line element is undefined, and there is no possibility of a black hole, which is alleged to occur in Hilbert's "Schwarzschild's solution" with infinitely dense singularity at $\mathrm{r}=0$ and event horizon at $\mathrm{r}=2 \mathrm{~m}$.

## Nomenclature

$\mathrm{a}^{\prime}, \mathrm{b}^{\prime}$ Derivative of $\mathrm{a}, \mathrm{b}$ with respect to r
$g_{\mu \nu}, g^{\mu \nu} \quad$ Metric tensor
$K=\frac{8 \pi G}{c^{4}}$, G is gravitational constant and
c is speed of light
$\mathrm{m} \quad$ Mass of the gravitating particle
P Fluid pressure
$R_{\mu \nu} \quad$ Ricci tensor
R Scalar curvature
$r \quad$ Radius of the sphere
$T_{v}^{\mu} \quad$ Energy tensor of matter
$u^{i} \quad$ Unit time like flow vector of the fluid
$\rho \quad$ Matter density
$\Lambda \quad$ Cosmological constant
$\delta_{v}^{\mu} \quad$ Kronecker delt

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